Reductive Groups and GIT

Throughout, let k be a field of char 0,
(in char $p \neq 0$, reductive groups aren't too interesting).

Quotients

Often it is useful to consider the following situation: Let G be a group object acting on an object X.

When can we form the quotient object X/G?

What should the quotient be?

<u>Ex</u>: In Set, we just take X/G to be the set of orbits. Then G acts trivially on X/G and the fibre of each point in X/G is an orbit in X.

__Ex__: In topology, we can endow the set of orbits with the quotient topology, so that

$X \to X/G$ is continuous.

__Ex__: In smooth manifolds, G is now a Lie Group and X a smooth manifold. Then without restricting the action of G, the set of orbits will __not__ be a smooth manifold (usually because non-closed orbits give a non-Hausdorff quotient).

We restrict to G acting freely and properly and everything works out.

Now we look int the category of ~~affine~~ schemes of finite type over k (or varieties).

Let G be an affine algebraic group (that is, affine scheme with group structure given by algebraic morphisms).

If G acts on X (f.t over k) then when can we form X/G? What should X/G be?

To understand this, let's abstract what a quotient is.

A <u>categorical quotient</u> of X by G is a morphism $\pi : X \to Y$ s.t.

$$\begin{tikzcd}
G \times X \arrow[r, "\sigma"] \arrow[d, "pr"'] & X \arrow[d, "\pi"] \arrow[ddr, bend left] & \\
X \arrow[r, "\pi"'] \arrow[drr, bend right] & Y \arrow[dr, dashed, "\exists !"] & \\
& & Z
\end{tikzcd}$$

commutes.

If Y exists we denote it by $X /\!/ G$.

Note that Y ($X/\!/G$) is not necessarily an orbit space, (i.e. fibres of π may not be closed orbits).

In this sense, a categorical quotient is different to a <u>geometric</u> quotient, X/G, which is an orbit space.

When does $X /\!\!/ G$ exist?

Here is a recipe when $X = \operatorname{Spec} A$ is affine.

G acts on $X = \operatorname{Spec} A$ which induces an action on the finitely generated k-algebra $A = \mathcal{O}_X(X) := \mathcal{O}(X)$.

Explicitly, $g \cdot f(p) = f(g^{-1}p)$.

Let $\mathcal{O}(X)^G = \{ f \in \mathcal{O}(X) \mid g \cdot f = f \}$ be the subalgebra of G-invariant elements.

Then define $Y = \operatorname{Spec} \mathcal{O}(X)^G$. Then we get a G-invariant map $X \xrightarrow{\pi} Y$ s.t. any other G-invariant morphism $X \to Z$ factors through π.

It seems that $Y = X /\!\!/ G$. However, Y is not necessarily of finite type ($\mathcal{O}(X)^G$ is not necessarily finitely generated k-algebra).

Hilbert's 14^{th} Problem: A f.g k-algebra

When is A^G finitely generated?

This is classical invariant theory. Hilbert proved for $G = GL_n$ over $\mathbb{C}$.

- Nagata showed false in general (by taking copies of $\mathbb{G}_a$)
- Nagata proved true for a large class of groups, "Reductive groups".

"Reductive $\longrightarrow$ Reynolds Operator $\longrightarrow$ Reduce to A polynomial algebra + Hilbert basis."

Reductive Groups

For $k = \bar{k}$ char 0, and G a smooth affine algebraic group, the notions of Reductive and linearly reductive coincide. We give the latter definition as it is nicer.

Def: An affine algebraic group G is linearly reductive if every finite dimensional linear representation $\rho : G \longrightarrow GL(V)$ is completely reducible, i.e. decomposes as a direct sum of irreducible ~~par~~ reps.

Ex: All finite groups, GL_n, SL_n, PSL_n and G_m are linearly reductive.

Ex: The additive group $G_a = \operatorname{Spec} k[t]$ is not linearly reductive.

Thus we see that if G is reductive, then $X /\!/ G$ ~~is~~ exists and we call it the GIT quotient.

Reductive groups were not known to Hilbert as the representation theory had not yet been fully developed.

Comments on Reductive groups

- Very interesting in their own right!
- Chevalley showed that the classification of reductive groups does not depend on k $(k=\bar{k})$.
- Recommend Milne's book Algebraic Groups.

Geometric Invariant Theory

We constructed the GIT quotient $X/\!\!/G$ when G is reductive. But $X/\!\!/G$ is not always a geometric quotient X/G, i.e. orbits are closed.

Ex: Consider $\mathbb{G}_m = \operatorname{Spec} k[t,t^{-1}]$ acting on $X = \mathbb{A}^2 = \operatorname{Spec} k[x,y]$ by $t\cdot(x,y) = (tx, t^{-1}y)$. The orbits are

- Conics $\{(x,y) \mid xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 - \{0\}$
- punctured x-axis
- punctured y-axis
- the origin.

The origin and conics are closed orbits, but the punctured axis are not, and since they contain the origin in their closure, then these orbits will be identified in $X /\!/ G = \cancel{\mathrm{Sp}}\ \operatorname{Spec} k[xy]$. So $X /\!/ G \neq X/G$. (here the quotient is $\pi: (x,y) \to xy$).

However, Mumford's innovation was the notion of stability:

<u>Def</u>: A point $x \in X$ is said to be <u>stable</u> for G if

1) The orbit $G \cdot x \subseteq X$ is closed
2) $\dim G_x = 0$ (the stabiliser subgroup).

Let X^s be the stable points.

<u>Theorem</u>: Let G be reductive acting on an affine scheme X and let $\pi: X \to X /\!/ G = Y$ be the GIT quotient. Then $X^s \subset X$ is open, G-invariant, and $\pi|_{X^s}$ is a geometric quotient.

Remarks: • The property $\dim G_x = 0$ ensures X^s is open, which is important, and if X is a variety, then X^s is dense.

• Mumford also "compactified" X^s by means of his notion of semi-stability.

Mumford's GIT and stable locus are key for constructing moduli spaces in algebraic geometry.

Ex: let $G = \mathbb{G}_m$ and $X = \operatorname{Spec} k[x,y] = \mathbb{A}^2$ again.

Then origin and conics are closed orbits, but $\dim G \cdot \{0\} = \dim \{0\} = 0 < \dim \mathbb{G}_m = 1$.

Hence $\dim G_{(0,0)} > 0$, so $(0,0)$ is not stable.

Hence $X^s = \{(x,y) \in \mathbb{A}^2 \mid xy \neq 0\} = X_{xy}$. ∎