Étale Coverings and Galois Categories

George Mitchell
City University of New York
Course: Algebra II *Instructor:* Prof. L. Szpiro

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Introduction

In these notes we will discuss étale coverings of schemes, the algebraic fundamental group, and will also see how the "Galois Theory" of fields, covering spaces and schemes are all examples of Galois Categories.

One aim of these notes is to introduce the notion of the "algebraic" fundamental group for schemes, so we begin by giving an example of when the "topological" fundamental group of a scheme gives undesirable results, and use this to motivate what is to follow. [3].

Motivation

Consider $\mathbb{C}$ with the usual metric topology and let $\mathbb{C}^* = \mathbb{C} - \{0\}$ be the punctured complex plane. It is a well known result from topology that $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$. Now endow $\mathbb{C}^*$ with the cofinite topology (the topology where the closed sets are precisely the finite sets). Then in this topology, a map $f : [0,1] \longrightarrow \mathbb{C}^*$ is continuous if and only if $f^{-1}(\{x\})$ is closed for all $x \in \mathbb{C}^*$. Compared to being continuous in the metric topology, this is very weak. In fact, if $\gamma, \psi : [0,1] \longrightarrow \mathbb{C}^*$ are two loops based at 1, then we may define a homotopy:

$$H(s,t) = \begin{cases} \gamma(s) & t = 0 \\ \psi(s) & t = 1 \\ 1 & s = 0, 1 \\ f(s,t) & \text{else} \end{cases}$$

where $f(s,t)$ is an arbitrary bijection from what remains of the domain into $\mathbb{C}^*$. In the cofinite topology, this homotopy is continuous and thus $\pi_1(\mathbb{C}^*) \cong 0$. Thus we see that the "topological" fundamental group gives undesirable results when the topology is sufficiently different from the metric topology. The cofinite topology on $\mathbb{C}^*$ is precisely the Zariski topology, and so when we consider schemes with the Zariski topology, this discrepancy will reoccur.

Defining the fundamental group as homotopy classes of loops is where the problem lies. Fortunately, we can compute the fundamental group using covering space theory, where

$\pi_1(X) \cong Deck(\tilde{X}/X)$, the group of deck transformations of the universal cover $\tilde{X}$ of a space X. This lends itself to algebraic generalisation much better than homotopy, and this leads to the definition of étale coverings for schemes.

Schemes and Morphisms

In this section we define finite étale coverings of schemes. See [2] for more on schemes.

Definition 1: Let (X, F) and (Y, G) be schemes. A *morphism* from X to Y is a pair $(\phi, \phi^\sharp)$ such that

- $\phi : X \longrightarrow Y$ is continuous,
- $\phi^\sharp : G \longrightarrow \phi_*(F)$ is a morphism of sheaves on Y,
- $\phi_x : G_{\phi(x)} \longrightarrow F_x$ is a homomorphism of local rings with $\phi_x^{-1}(\mathfrak{m}_F) = \mathfrak{m}_G$.

If $(X, \mathcal{O}_X)$ is a scheme and $U \subseteq X$ is open, then the inclusion map $i : (U, \mathcal{O}_X|_U) \longrightarrow (X, \mathcal{O}_X)$ is a morphism, called an *open immersion*.

We are now ready for our analogue of "covering" to schemes. The following definition is not the standard seen in the literature and is due to Lenstra [1]. It is more hands on and will be better for our purposes.

Definition 2: A morphism of schemes $f : Y \longrightarrow X$ is called a *finite étale covering of* X if the following hold:

- for all $U \subset X$ affine, $f^{-1}(U)$ is affine in Y,
- if $U = \text{Spec}A$ affine in X, and $f^{-1}(U) = \text{Spec}B$ in Y, then B is a finitely generated, projective separable A-algebra.

Recall that B is a separable A-algebra if the map $\phi : B \longrightarrow Hom_A(B, A),\ \phi(x)(y) = Tr(xy)$ is an isomorphism.

Suppose $f : Y \longrightarrow X,\ g : Z \longrightarrow X$ are finite étale covers of X, then a *morphism of covers* is a morphism $h : Y \longrightarrow Z$ such that $f = gh$. Thus we may speak of $\mathbf{FEt_X}$, the category of all finite étale coverings of a given scheme X.

Why is this definition a good analogue of "covering maps" and in what sense does it give an "algebraic" fundamental group? We first recall the following theorem from topology:

Theorem 1: Let X be a connected topological space. Then there is an equivalance of categories between finite covers of X and $\hat{\pi_1}(X)$-sets, i.e. finite sets on which $\hat{\pi_1}(X)$ acts continuously. Note $\hat{\pi_1}$ denotes the profinite completion of π_1.

We prove the following main theorem which makes the analogy between topology and étale clear:

Theorem (Main): Let X be a connected scheme. Then there exists a profinite group π, unique up to isomorphism, such that there is an equivalance of categories between $\mathbf{FEt_X}$ and π-sets, i.e. finite sets with a continuous action of π.

To prove this theorem we will need the notion of a Galois Category with Fundamental Functor. This notion generalises and combines the Galois theories of fields, covering spaces and schemes.

Galois Categories and Fundamental Functor

From now on we follow the path of Lenstra [1].

Definition 3: Let $\mathcal{C}$ be a category and $\mathcal{F}$ a covariant functor from $\mathcal{C}$ to **Set**. Then $(\mathcal{C}, \mathcal{F})$ is called a *Galois category with Fundamental functor* if the following properties hold:

A1. $\mathcal{C}$ has a terminal object, and the fibred product of any two objects over a third exist in $\mathcal{C}$.

A2. $\mathcal{C}$ has an initial object, the sum of two objects exist in $\mathcal{C}$, and for any object X, the quotient X/G exists in $\mathcal{C}$, where G is a finite subgroup of $Aut(X)$.

A3. Any morphism h in $\mathcal{C}$ can be decomposed as $h = f \circ g$ where g is an epimorphism and f is a monomorphism. Moreover, any monomorphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is an isomorphism of X with a direct summand of Y.

A4. $\mathcal{F}$ preserves the terminal object and commutes with fibred products.

A5. $\mathcal{F}$ commutes with finite sums, quotients by a finite group of automorphisms and sends epis to epis.

A6. If $\mathcal{F}(f)$ is an isomorphism then f is an isomorphism.

From the above it can be shown that $(\mathbf{Set}, Id)$ is a Galois category. We will see other examples later. Given a Galois category, we can define the "fundamental group" for this category:

Definition 4: Let $(\mathcal{C}, \mathcal{F})$ be a small (objects of $\mathcal{C}$ form a set) Galois category. The *fundamental group* of $(\mathcal{C}, \mathcal{F})$ is $Aut(\mathcal{F}) = \{\eta : \mathcal{F} \longrightarrow \mathcal{F} \mid \eta \text{ is invertible}\}$.

The profinite group π from the main theorem will be $Aut(\mathcal{F})$ for some Galois category and fundamental functor F yet to be defined. We know $Aut(\mathcal{F})$ is profinite because it is a closed subgroup of $\prod_X S_{\mathcal{F}(X)}$, where $S_{\mathcal{F}(X)}$ is the group of permutations of $\mathcal{F}(X)$. This is a product of profinite (actually finite) groups and so is profinite.

The profinite group $Aut(\mathcal{F})$ acts continuously on $\mathcal{F}(X)$ for each object X, so we define a functor $H : \mathcal{C} \longrightarrow Aut(\mathcal{F})$-sets where $H(X) = \mathcal{F}(X)$ but with the continuous $Aut(\mathcal{F})$ action.

The main theorem to be proved is a corollary of the following theorem due to Grothendieck:

Theorem 2: The functor H defined above is an equivalence of categories and $Aut(\mathcal{F})$ is unique up to inner automorphism.

Our proof of the main theorem will be to show that $\mathbf{FEt_X}$ is a Galois category, though we have not defined the fundamental functor for this category. First we illustrate two important examples.

Example 1: Let X be a connected topological space and let $\mathbf{Cov_X}$ be the category of finite covers of X. Let $x \in X$. We can define a functor $\mathcal{F}_x : \mathbf{Cov_X} \longrightarrow \mathbf{Set}$ where $\mathcal{F}_x(p : Y \to X) = p^{-1}(x)$. Then $(\mathbf{Cov_X}, \mathcal{F}_x)$ is a Galois category and $Aut(\mathcal{F}_x) \cong \hat{\pi}_1(X, x)$. Applying Theorem 2 in this situation gives a generalisation of the Galois theory of covering spaces.

Example 2: Let k be a field of characteristic zero and let $\mathbf{SAlg_k}$be the category of free separable k-algebras. Define a functor $\mathcal{F}_k : \mathbf{SAlg_k} \longrightarrow \mathbf{Set}$ by $\mathcal{F}_k(L) = Hom_k(L, \bar{k})$ where $\bar{k}$ is a fixed algebraic closure of k. Then $(\mathbf{SAlg_k}, \mathcal{F}_k)$ is a Galois category and $Aut(\mathcal{F}_k) \cong Gal(\bar{k}/k)$, the absolute Galois group of k. Applying Theorem 2 in this case gives a generalisation of the Galois theory of fields.

We will now prove the main theorem by showing that $\mathbf{FEt_X}$ is a Galois category.

Properties of $\mathbf{FEt_X}$

To show that $\mathbf{FEt_X}$ satisfies the axioms of a Galois category, we require two notions. The first will allow us to prove properties relating to the category and the second will allow us to define the fundamental functor. For all of the following see [1].

Degree

Let A be a ring and P a finitely generated projective A-module. For $\mathfrak{p} \in \text{Spec}A$ we know that $P_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module of finite rank. Thus we can define a function $Rank(P) : \text{Spec}A \longrightarrow \mathbb{Z}$ where $Rank(P)(\mathfrak{p}) = rank_{A_\mathfrak{p}}(P_\mathfrak{p})$. Since $Rank(P)$ is locally constant, we know that it is indeed continuous.

Let $f : Y \longrightarrow X$ be a finite étale cover. For each open affine set $\text{Spec}A$ in X, with $f^{-1}(\text{Spec}A) = \text{Spec}B$, there is a continuous rank function $[B : A] : \text{Spec}A \longrightarrow \mathbb{Z}$, where $[B : A] = Rank(B)$ as above. We can patch these local rank functions together to obtain a global continuous rank function $[Y : X] : X \longrightarrow \mathbb{Z}$, called the *degree* of f.

We end this subsection by listing without proof some properties of the degree:

- For all $n \in \mathbb{Z}$, the set $\{x \in X \mid [Y : X](x) = n\}$ is both open and closed.
- The degree $[Y : X]$ is constant on connected components of X.
- $Y = \emptyset$ if and only if $[Y : X] \equiv 0$.
- If $Y \longrightarrow X$ and $Z \longrightarrow X$ are two finite étale covers, with $[Y : X] = [Z : X]$, and $h : Y \longrightarrow Z$ is a surjective morphism of covers, then h is in fact an isomorphism.

Geometric Points

The notion of geometric points of a scheme will allow us to define the fundamental functor for $\mathbf{FEt_X}$.

Definition 5: A *geometric point* of a scheme X is a morphism $x : \mathrm{Spec}\Omega \longrightarrow X$ where Ω is an algebraically closed field.

Note since Spec$k$ for any field k is just a point, we may refer to the above morphism as a "point" in X, by reference to the image of the morphism.

Let k be a field of characteristic zero. Then as a scheme, Spec$k$ is a point whose sheaf is k. Suppose $f : Y \longrightarrow \mathrm{Spec}k$ is a finite étale covering. By definition, we know that $f^{-1}(\mathrm{Spec}k) = \mathrm{Spec}B$, where B is a finitely generated, separable projective k-algebra. Since projective modules are direct summands of free modules, we can conclude that Y corresponds to a free separable k-algebra of finite rank. Thus we have an equivalence of categories between $\mathbf{FEt_{Spec\Omega}}$ and $\mathbf{SAlg_k}$.

By Example 2 above we know that $\mathbf{SAlg_k}$ is equivalent to the category of $Gal(\bar{k}/k)-$sets. But if k is algebraically closed, then $Gal(\bar{k}/k)$ is trivial and so in this case $\mathbf{SAlg_k}$is equivalent to **Set**. Combining this with the notion of geometric points, we are able to define the fundamental functor for $\mathbf{FEt_X}$as follows:

Let X be a connected scheme and let $x : \mathrm{Spec}\Omega \to X$ be a geometric point. If $f : Y \longrightarrow X$ is a finite étale covering, then $Y \times_X \mathrm{Spec}\Omega \longrightarrow \mathrm{Spec}\Omega$ is also a finite étale covering, and thus we get a functor $H_x(-) : \mathbf{FEt_X} \longrightarrow \mathbf{FEt_{Spec\Omega}}$ where $H_x(Y) = Y \times_X \mathrm{Spec}\Omega$. By the previous discussion, there is an equivalence $J : \mathbf{FEt_{Spec\Omega}} \longrightarrow \mathbf{Set}$. Thus we define the fundamental functor of $\mathbf{FEt_X}$to be $\mathcal{F}_x = J \circ H_x$.

Proof of Main Theorem

We prove the main theorem by showing that $(\mathbf{FEt_X}, \mathcal{F}_x)$ is a Galois category.

Proof of Main Theorem:

A1: The morphism $Id : X \longrightarrow X$ is terminal in $\mathbf{FEt_X}$. We can use the fibred product for schemes (whose existence is proved in [2]) as a fibred product in $\mathbf{FEt_X}$. See [1] also.

A2: The morphism $\emptyset \longrightarrow X$ is initial in $\mathbf{FEt_X}$. Existence of X/G for a finite group of automorphisms of X is shown in [1], and is beyond our scope.

Let $f_1 : Y_1 \longrightarrow X$ and $f_2 : Y_2 \longrightarrow X$ be finite étale covers of X. We define $f : Y_1 \sqcup Y_2 \longrightarrow X$ by $f((y,i)) = f_i(y)$. Let Spec$A$ be an affine open set in X. Then by assumption we know that $f_1^{-1}(\mathrm{Spec}A) = \mathrm{Spec}B$ and $f_2^{-1}(\mathrm{Spec}A) = \mathrm{Spec}C$, where B and C are both finitely generated, separable projective A-algebras. Then $f^{-1}(\mathrm{Spec}A) = \mathrm{Spec}B \sqcup \mathrm{Spec}C = \mathrm{Spec}(B \times C)$. Since B, C are both finitely generated, projective separable A-algebras, so is $B \times C$. Thus sums exist in $\mathbf{FEt_X}$.

A3: Let $h : Y \longrightarrow Z$ be a morphism of covers. Partition $Z = Z_0 \sqcup Z_1$ where $Z_0 = \{z \in Z \mid [Y : Z](z) = 0\}$, and $Z_1 = Z \backslash Z_0$. Then by the properties of degree we know that $h^{-1}(Z_0) = \emptyset$ and so h factors as $Y \to Z_1 \to Z_1 \sqcup Z_0$ which is an epimorphism followed by a monomorphism. The second assertion of A3 follows in a similar way.

A4: The functor $\mathcal{F}_x$ preserves the terminal object and commutes with the fibred product because the base change $- \times_X \text{Spec}\Omega$ does.

A5: Again the functor $\mathcal{F}_x$ commutes with sums, quotients and sends epis to epis because the base change does.

A6: Let X be connected and $f : Y \longrightarrow X$ be finite étale. We know that $[Y : X]$ is constant on X. Then $[Y : X] = [H_x(Y) : \text{Spec}\Omega]$. Thus it follows that $|\mathcal{F}_x(Y)| = [Y : X]$.

Let $h : Y \longrightarrow Z$ be a morphism of covers and suppose that $\mathcal{F}_x(h) : \mathcal{F}_x(Y) \longrightarrow \mathcal{F}_x(Z)$ is a bijection. We can decompose h as $Y \to Z_1 \to Z_0 \sqcup Z_1$ as before, where $Y \to Z_1$ is surjective. Then $\mathcal{F}_x(Z_1) \to \mathcal{F}_x(Z_0) \sqcup \mathcal{F}_x(Z_1)$ is surjective since it preserves epimorphisms. But then $\mathcal{F}_x(Z_0) = \emptyset$, and so $[Z_0 : X] = 0$. Thus $Z_0 = \emptyset$. Hence $Z = Z_1$ and $h : Y \longrightarrow Z$ is surjective. Then $[Y : X] = [Z : X]$ and by properties of the degree, h is an isomorphism.

□

Algebraic Fundamental Group

Knowing that $\mathbf{FEt_X}$is a Galois category, allows us to define the "algebraic" fundamental group for a scheme.

Definition 6: Let $(\mathbf{FEt_X}, \mathcal{F}_x)$ be as above for a given connected scheme X and a geometric point $x \in X$.. Then we define the *algebraic fundamental group* of X at x, denoted as $\pi_1^{\text{alg}}(X, x)$, to be $Aut(\mathcal{F}_x)$, as in the main theorem.

As a final remark, it can be shown that $\pi_1^{\text{alg}}(\mathbb{C}^*) = \hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$, thus salvaging the fundamental group from the issues that were outlined in the beginning. See [4].

References

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[4] Tamas Szamuely. *Galois Groups and Fundamental Groups.* Cambridge University Press, 2009.