In the time that is left, I would like to discuss the GAGA analogs to non-archimedean geometry.

Formal GAGA

We use formal schemes to construct an analogous analytification of X but over other fields than $\mathbb{C}$.

Quick Intro to Formal Schemes

Let A be a Noetherian ring and $I \subset A$ an ideal. Give A the I-adic topology whereby $\{I^n\}_n$ form a basis of n'hoods of 0.

Def: Call A an adic-ring if it is complete and separated in this topology, and I is the ideal of definition.

Let $X_n = A/I^n$. Then each of $\operatorname{Spec} X_n$ have the same underlying space, namely $\operatorname{Spec} X_1 = \operatorname{Spec} A/I$.
On each X_n we have a sheaf of rings $\mathcal{O}_n$ which form an inverse system. Taking the limit we obtain

$(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ a topologically locally ringed space, where $\mathfrak{X} = |\operatorname{Spec} A/I|$ and $\mathcal{O}_{\mathfrak{X}} = \varprojlim \mathcal{O}_n$.

This is called the formal spectrum of A and is denoted Spf A.

Def: A ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ that is isomorphic to Spf A is called an affine formal scheme. If its locally isomorphic then call it a formal scheme.

Key Construction: X locally Noetherian and X' a closed subscheme with ideal $\mathcal{I}$.
Then locally X looks like Spec A with an ideal I of A being X'. We can take the Spf A and then glue to get a formal scheme $(\hat{X}, \mathcal{O}_{\hat{X}})$, called the formal completion of X ~~by~~ along X'.

This is going to be the analytification.

Theorem (GFGA): Let A be a Noetherian ring complete w.r.t to an ideal I. Let X be a proper A-scheme and let X' be the locus of I in X. Let $\mathfrak{X} = (\hat{X}, \mathcal{O}_{\hat{X}})$ be the formal completion along X'.

Then the functor $\mathcal{F} \mapsto \hat{\mathcal{F}}$ from coherent $\mathcal{O}_X$-Modules to coherent $\mathcal{O}_{\mathfrak{X}}$-modules is an equivalence of categories.

Rmk: $\hat{\mathcal{F}} = \varprojlim_n \left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I}^{n+1} \right)$.

The formal GAGA (GFGA) again can be proved the same way as classical GAGA after showing analogous statements to Theorem I and Theorem II.

Again, proving the formal Cartan Thm A is more difficult than the algebraic case.

This is slightly easier than the Complex Analytic Case because to give a coherent module on $\mathfrak{X}$ is to give compatible coherent modules on the pieces X_n.

Applications of GFGA

<u>Lifting Problem</u>: Let $S = \operatorname{Spec} A$ be a local noetherian affine scheme. Let $k = A/\mathfrak{m}$ and suppose X_0 is a scheme of finite type over $\operatorname{Spec} k$.

Can we find X, finite type and flat over $\operatorname{Spec} A$ such that X_0 is the special fibre of $X \to \operatorname{Spec} A$?

<u>Grothendieck's strategy</u> is:

(1) Lift X_0 inductively to a system $\{X_n\}$ where X_n is flat and finite type over $\operatorname{Spec} A/\mathfrak{m}^{n+1}$ with some compatibility. Cohomology of X_0 contains obstructions to lifting.

(2) From $\{X_n\}$ we get a formal scheme $\mathfrak{X}$ over $\operatorname{Spf} \hat{A}$ which is flat and finite type.

(3) Assuming X_0 is projective, then GFGA gives a projective scheme X such that $\mathfrak{X} \simeq \hat{X}$. X is over $\operatorname{Spec} \hat{A}$.

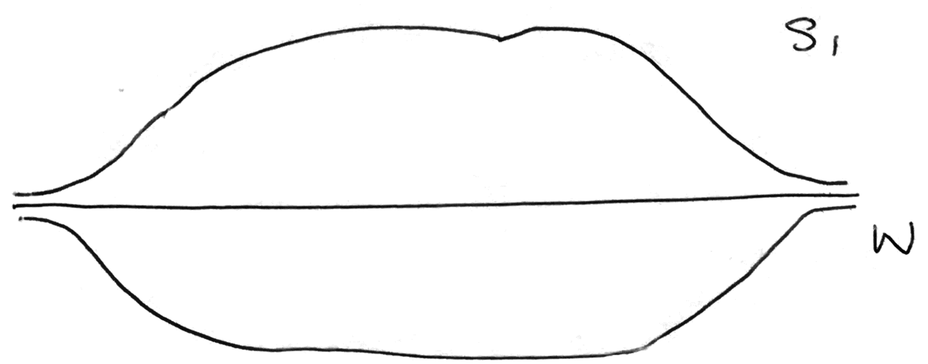
(4) In general one cannot descend X to be over $\operatorname{Spec} A$.

Another application

<u>Inverse Galois Problem</u>: Let R be a normal complete local domain, not a field. Let $K = \mathrm{frac}(R)$ and G a finite group. Then there is a G-Galois branched cover of $\mathbb{P}^1_K$, moreover it is regular.

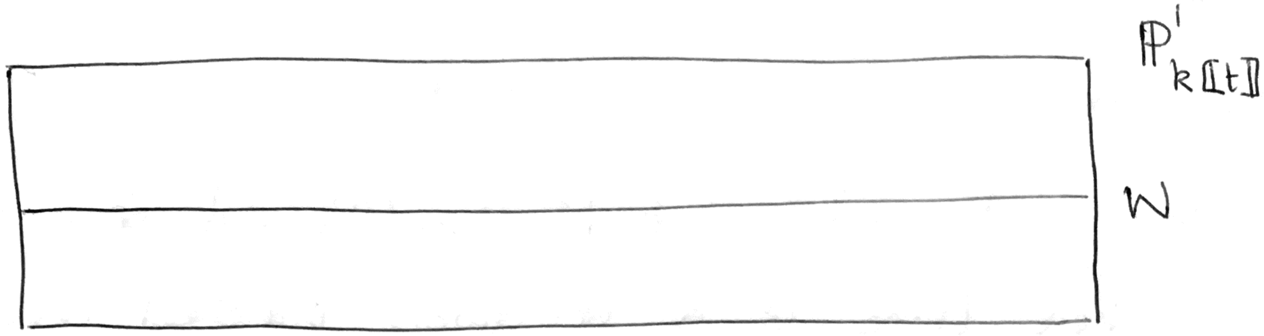
This is proven using formal patching, analogous to the analytic patching. This works because we can view formal spectra as formal n'hoods of holomorphic functions.

<u>Ex</u>: Let $V = \mathbb{A}^2_k$ and let $W = \mathbb{A}^1_k$ defined by $I = (t)$. The ring of formal holomorphic functions along W is the completion of $A = k[x,t]$ by I, so $A_1 = k[x][[t]]$. Intuitively, $S_1 = \mathrm{Spec} A_1$ is a tubular n'hood ~~o~~f W that pinches at ∞:

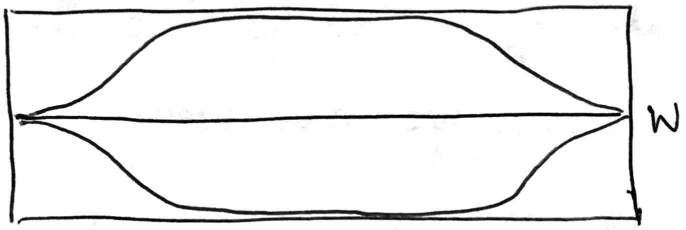


Because $x, x-t$ define points in S_1 but not $1-xt$.

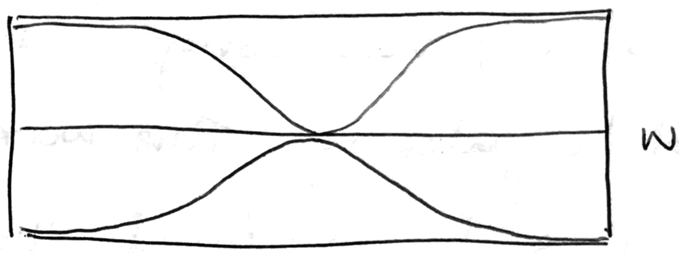
Using this idea, we can take



and cover it with two patches

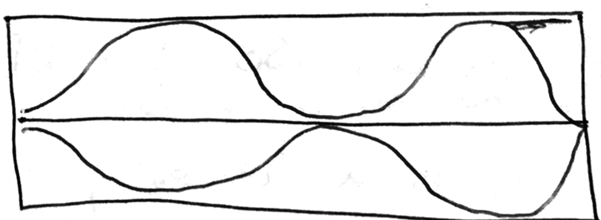


$S_1 = \operatorname{Spec} k[x][[t]]$



$S_2 = \operatorname{Spec} k[\frac{1}{x}][[t]]$

these are <u>formal</u> but not zariski. Their overlap is:



$S_0 = \operatorname{Spec} k[x, \frac{1}{x}][[t]]$

Like analytic patching, if we build a cover on the formal n'hoods S_1, S_2 that agree on S_0 then by GFGA we can find a algebraic scheme that realises the cover. Moreover these formal n'hoods are tighter than zariski n'hoods.