Rigid GAGA

Rigid Geometry was invented by Tate to bring complex analytic techniques to geometry over fields k, that are complete w.r.t. a non-archimedean ~~valuation~~ abs value.

His initial motivation was to study Elliptic curves with bad reduction.

The problem is that the topology on k induced from the valuation is totally disconnected, so that any notion of analytification continuation will not work.

Ex: The function $f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

is analytic in $k = \mathbb{Q}_p$, but unlike over $\mathbb{C}$, such functions are not determined by their value on open sets. So no identity theorem will exist.

Tate's idea was to limit the allowable functions and open sets. This leads to a Grothendieck topology.

Rigid Spaces

For brevity and ease of notation, we will consider the 1-dimensional case.

Analytic functions should be given locally by convergent power series on disks, so that is our starting point:

Def: The Tate Algebra is

$$k\{t\} := T := \left\{ \sum a_n t^n \in k[[t]] \mid |a_n| \to 0 \text{ as } n \to \infty \right\}$$

These are the power series that converge on

$$\mathbb{B} = \{ z \in k \mid |z| \leq 1 \}.$$

There are two analytic properties:

(1) T is a k-Banach algebra w.r.t to the Gauss Norm:

$$\|f\| := \max_i |a_i|$$

(2) The Maximum Modulus Principle holds:

$$\|f\| = \max_{z \in \mathbb{B}} |f(z)|.$$

We want T to be the "coordinate ring" of $\mathbb{B}$. Using the Weierstrass Theory, one can show the following algebraic facts that show T behaves like $k[x]$:

<u>Theorem</u> (i) T is noetherian and a UFD

(ii) T is Jacobson, so for any ideal $I \subset T$,

$$\sqrt{I} = \bigcap_{I \subset m} m.$$

(iii) Every ideal of T is closed

(iv) For maximal $m \subset T$, the residue T/m is finite over k.

This last property is the analogue of a Nullstellensatz.

These properties show that $\operatorname{maxSpec} T$ is a good algebraic model for $\mathbb{B}$. Property (ii) shows why taking maxSpec is sufficient. It is also functorial in this case.

- What distinguishes T from usual k-analytic manifolds is that T will have many non-k-rational points, just like in schemes. This will be useful.

Def: A k-algebra A is called affinoid if $A \simeq T^n/I$ for some $n \in \mathbb{N}$ and ideal I.

- A is noetherian and Jacobson
- A is a k-banach algebra w.r.t to any residue norm.
- The topology on A is independent of the residue norm.
- Any hom between ~~k~~ affinoids is continuous.

Set $M(A) = \max \operatorname{Spec} A$.

Then this is functorial in A by prop (iv) and we can define

$$|f|_{\sup} := \sup_{x \in M(A)} |f(x)|$$

where we view $f \in A$ as a function on $M(A)$ by evaluating to the residue class.

- The MMP says that

$$|f|_{\sup} = \max_{x \in M(A)} |f(x)|.$$

The key idea of Tate is to globalise the affinoids, but preserve nice properties, we must restrict the open sets, and moreover, the kinds of coverings by open sets.

There are 3 types of domain, Weierstrass, Laurent and Rational. But we will only need the rational.

<u>Def</u>: Let $X := \mathrm{Max\,Spec}\, A$, A affinoid. Let $g, f_1, \ldots, f_m \in A$ have no common zero (i.e. $\langle g, f_i \rangle = A$). The following:

$$X\left(\frac{f_\cdot}{g}\right) := \{ x \in X : \max_i |f_i(x)| \leq |g(x)| \}$$

are called <u>rational subdomains</u>. They form a basis for a topology on X, called the <u>canonical topology</u>.

The canonical topology on X is hausdorff but also totally disconnected, so it is still not helpful.

What should the coordinate ring of $X\left(\frac{f_\cdot}{g}\right)$ be?

Theorem: The k-algebra $A\langle\frac{f_\cdot}{g}\rangle := \frac{A\{t_1,\ldots,t_m\}}{\langle gt_i - f_i\rangle}$

is affinoid, and the map

$$\operatorname{max\,Spec} A\langle\frac{f_\cdot}{g}\rangle \longrightarrow \operatorname{max\,Spec} A = X$$

induced by $A \to A\langle\frac{f_\cdot}{g}\rangle$ is a homeomorphism onto $X\langle\frac{f_\cdot}{g}\rangle$.

Rmks: • $A\{t_1,\ldots,t_m\}$ is the Tate algebra over A

• $A\langle\frac{f_\cdot}{g}\rangle$ can be characterised by a universal mapping property in terms of $X\langle\frac{f_\cdot}{g}\rangle$, and so is intrinsic to $X\langle\frac{f_\cdot}{g}\rangle$.

This allows one to define a presheaf $\mathcal{O}_X$ on the rational subdomains in X by

$$\mathcal{O}_X(X(\tfrac{f_\cdot}{g})) := A\langle\frac{f_\cdot}{g}\rangle.$$

Tate's Acyclicity Theorem: If $Y, Y_1, \ldots, Y_r \subseteq X$ are rational subdomains such that $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r$, then $\mathcal{O}_X$ satisfies the sheaf property for that covering:

$$0 \longrightarrow \mathcal{O}_X(Y) \longrightarrow \prod_i \mathcal{O}_X(Y_i) \rightrightarrows \prod_{i,j} \mathcal{O}_X(Y_i \cap Y_j)$$

is exact.

This shows that analytic continuation will work so long as we take finite unions of rational subdomains.

We enlarge the situation formally:

* A subset $\mathcal{U} \subseteq X$ is called <u>admissible</u> if there are rational subdomains $U_i \subseteq X$ such that

  i) $\mathcal{U} = \bigcup_i \mathcal{U}_i$

  ii) for any map $\alpha : Y = \text{maxSpec} B \longrightarrow X = \text{maxSpec} A$ induced by $A \to B$ with $\text{im}(\alpha) \subset \mathcal{U}$, the covering $\bigcup_i \alpha^{-1}(U_i)$ of Y has a finite subcovering.

* Let V and V_j be admissible opens of X s.t. $V = \bigcup_j V_j$.
  The covering is called <u>admissible</u> if for any map $\alpha : Y \to X$ with $\text{im}(\alpha) \subseteq V$, the covering $\bigcup \alpha^{-1}(V_j)$ of $\forall\, Y$ ~~has a~~ can be refined into finite subcover by rational subdomains.

The admissible opens and admissible coverings form a Grothendieck topology on X and using Tate's Theorem, $\mathcal{O}_X$ becomes a sheaf. Thus $(X, \mathcal{O}_X)$ becomes a G-ringed Space, <u>affinoid variety</u>.

Let's see an example of how this restriction solves the issues of connectedness.
In particular, the closed unit disk becomes connected.

Ex: Let $X = \operatorname{maxSpec} k\{t\} =$ ~~[illegible]~~ the closed unit disk.

By construction we have $\mathcal{O}_X(X) = k\{t\}$

The "unit circle" $V = \{x \in X \mid |t(x)| = 1\}$ is a rational subdomain since $V = X(\frac{1}{t})$.

The subset $\mathcal{U} = \{x \in X \mid |t(x)| < 1\}$ is also admissible open:

choose $\varepsilon = |\pi|$ with $0 < \varepsilon < 1$ $(\pi \in k^\times)$ and set

$$\mathcal{U}_n := X\left(\frac{t^n}{\pi}\right) = \{x \in X \mid |t(x)| \leq \varepsilon^{1/n}\} \quad \forall\ n \in \mathbb{N}.$$

These are rational subdomains and $\mathcal{U} = \bigcup \mathcal{U}_n$.

Let $\alpha: Y = \operatorname{maxSpec}(B) \to X$ be a morphism of affinoids such that $\operatorname{Im}(\alpha) \subseteq \mathcal{U}$, then the MMP gives

$$\max_{y \in Y} |t(\alpha(y))| < 1 \quad \text{and so there is } 0 < \alpha < 1$$

s.t. $|t(\alpha(y))| \leq \alpha \ \forall\ y \in Y$. Then for n_0 large enough such that $\alpha < \varepsilon^{1/n_0} < 1$, we have that

$\alpha(Y) \subset \mathcal{U}_{n_0}$. This gives the finite subcover.

It seems that $X = \mathcal{U} \cup V$ shows that the closed unit disk is still disconnected. However, this cover is not admissible:

Suppose it where. Take the identity map $X \to X$,
It would then follow that $\{\mathcal{U}, V\}$ has a finite subcover of X by ~~of~~ rational subdomains.

But any rational subdomain of X contained in $\mathcal{U}$ must be contained in some $\mathcal{U}_{n_0}$ by the MMP.

Thus such a refinement would show that X can be covered by V and $\mathcal{U}_{n_0}$.

By passing to a finite extension of k, we can find a point z of X with $|t(z)| = \varepsilon^{1/n_0+1}$ that is disjoint from $\mathcal{U}_{n_0} \cup V$. ↯.

<u>Def</u>: A <u>rigid analytic space</u> is a G-ringed space $(X, \mathcal{O}_X)$ with an admissible open covering such that each $(\mathcal{U}_i, \mathcal{O}_X|_{\mathcal{U}_i})$ is iso to an affinoid variety.

<u>Analytification</u>

Now let X be locally of f.t. over k. We define X^{an} locally and glue, so its enough to take $X = \mathrm{Spec}(A)$.

Set $X^{an} = \mathrm{maxSpec}(A)$. Also $A = \dfrac{k[X_1, \dots, X_n]}{I}$. Fix $c \in k$ with $|c| > 1$.

Set $\mathcal{U}_n = \{ x \in X^{an} \mid \max_j |X_j(x)| \leq |c|^n \}$. Then $X^{an} = \bigcup \mathcal{U}_n$.

Then it turns out that this is an admissible open cover.

This process gives a map of G-ringed spaces:

$$h : X^{an} \longrightarrow X$$

Such that

(1) h is bijective on closed points

(2) h induces on stalks $h^{\#} : \mathcal{O}_{X,\eta} \hookrightarrow \mathcal{O}_{X^{an},\eta}$ which is faithfully

(3) The pullback h^* is exact on $\mathcal{O}_X$-modules

(4) $h^* : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X^{an})$ is well defined.

Rigid GAGA: $h^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{an})$ is an equivalence when X is projective.

This is proved analogously, showing iso's on cohomology and an analogue of Cartan Thm A and B.

Applications

Rigid Geometry allows for "rigid patching" to build covers locally. Thus we can get

Theorem: Let k be a field complete w.r.t. a non-archimedean abs value. Let G be a finite group. Then there is an irreducible G-Galois cover (branched) $Y \to \mathbb{P}^1_k$.

Rigid GAGA yields similar results to formal GAGA

since Raynaud showed that the rigid space X^{an} is a generic fibre of the formal scheme $\mathfrak{X}$ (modulo blow ups.

This is why the two prove similar results.