Serre's GAGA and non-archimedean analogs

Let $(X, \mathcal{O}_X)$ be a scheme, locally of finite type over $\mathbb{C}$.

We can endow $X(\mathbb{C})$ with the complex metric topology and consider the sheaf of holomorphic functions on it. We denote the resulting locally ringed space as $(X^{an}, \mathcal{H}_X)$, called the <u>analytification</u> of X.

ASSUME NOW THAT X IS PROJECTIVE

$\boxed{\text{GAGA}}$: There is a bijective morphism of locally ringed spaces $h : X^{an} \longrightarrow X$ that induces isomorphisms $\forall\ i \geq 0$

$$\mathcal{H}^i(X, \overset{\mathcal{F}}{\cancel{\mathcal{O}_X}}) \xrightarrow{\sim} \mathcal{H}^i(X^{an}, \overset{h^*\mathcal{F}}{\cancel{\mathcal{H}_X}})$$

for $\mathcal{F} \in \mathrm{Coh}(X)$,
and an equivalence of categories

$$h^* : \mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(X^{an}).$$

We will give a breif overview of the proof of this fact.

The map $h : X^{an} \longrightarrow X$ is just the map on $X(\mathbb{C})$ and since X is locally given by polynomials (which are holomorphic), the map on sheaves is just "view polynomials as holomorphic functions" locally.

The map h has some key properties which we now list.

<u>Lemma</u>: $h : X^{an} \longrightarrow X$ satisfies

- bijective on closed points
- For $\eta \in X(\mathbb{C})$, the induced morphism $h^{\#} : \mathcal{O}_{X,\eta} \hookrightarrow \mathcal{H}_{X,\eta}$ is faithfully flat. (Isomorphism upon completion)
- The pullback $h^* : \mathrm{Mod}_{\mathcal{O}_X}(X) \longrightarrow \mathrm{Mod}_{\mathcal{H}_X}(X^{an})$ is an exact functor. Also $h^* \mathcal{O}_X = \mathcal{H}_X$.
- Therefore, $h^* : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X^{an})$ is well defined.

Note here that on both X and X^{an}, being <u>coherent</u> is the same as being locally finite pres, but for different reasons. If $\mathcal{F}$ is coherent on X, then locally $\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0$

and applying h^* gives $\mathcal{H}_X^m \to \mathcal{H}_X^n \to \mathcal{F}^h \to 0$

which is what it means to be coherent on X^{an}.

The equivalence of categories $h^* : Coh(X) \to Coh(X)$ follows formally once we know the following facts

<u>Theorem I</u>: $\forall i$, $H^i(X, \overset{\mathcal{F}}{\cancel{\mathcal{O}_X}}) \xrightarrow{\sim} H^i(X^{an}, \overset{h^*\mathcal{F}}{\cancel{\mathcal{H}_X}})$

<u>Theorem II</u>: For some $n \gg 0$, $\mathcal{F}(n)$ is generated by global sections. where $\mathcal{F}$ is either $Coh(X)$ or $Coh(X^{an})$.

The analytic part of Thm II is known as Cartan's Theorem A (or B) and was known to Serre already, therefore we take it as a black box.

Assuming these for now, lets prove the equivalence:

<u>proof</u>: We need to show h^* is fully faithful and essentially surjective.

Proof of GAGA equivalence

To prove equivalence of categories, we show that h^* is fully faithful and essentially surjective.

fully faithful: We need to show that for $\mathcal{F}, \mathcal{G}$ coherent, the map $\cancel{\phi: \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}_X}(}$

$$\phi : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}_X}(h^*\mathcal{F}, h^*\mathcal{G})$$

is bijective.

Note that these two sets are global sections of the following sheaves:

$$S = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}), \quad T = \mathcal{H}om_{\mathcal{H}_X}(h^*\mathcal{F}, h^*\mathcal{G})$$

There is a natural map $i : \cancel{S}\, h^*S \longrightarrow T$ inducing $i_* : H^0(X^{an}, h^*S) \longrightarrow H^0(X^{an}, T)$. Precomposing with the iso $\varepsilon : H^0(X, S) \xrightarrow{\sim} H^0(X^{an}, h^*S)$ gives the map

$$i_* \circ \varepsilon : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{H}_X}(h^*\mathcal{F}, h^*\mathcal{G})$$

which is an iso if i_* is an iso, which is if $i : h^*S \to T$ is an iso. This can be checked on stalks:

$$\begin{array}{c} (h^*S)_\eta = \mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathcal{F}_\eta, \mathcal{G}_\eta) \otimes_{\mathcal{O}_{X,\eta}} \mathcal{H}_{X,\eta} \\ \Big\downarrow \\ T_\eta = \mathrm{Hom}_{\mathcal{H}_{X,\eta}}(\mathcal{F}_\eta \otimes_{\mathcal{O}_{X,\eta}} \mathcal{H}_{X,\eta}, \mathcal{G}_\eta \otimes_{\mathcal{O}_{X,\eta}} \mathcal{H}_{X,\eta}). \end{array}$$

Since $\mathcal{O}_{X,\eta} \hookrightarrow \mathcal{H}_{X,\eta}$ is faithfully flat, then we pull the tensor out of the Hom and thus the above is an iso.

$\square$

We now prove essential surjectivity.

That is, if M is a coherent $\mathcal{H}_X$-module on X^{an}, then there is a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $h^*\mathcal{F} = M$.

First we can reduce to the case $X = \mathbb{P}^r_{\mathbb{C}}$ since $j : X \hookrightarrow \mathbb{P}^r_{\mathbb{C}}$, and we take $j_*\mathcal{F}$ on $\mathbb{P}^r_{\mathbb{C}}$.

This works because $j_*(h^*\mathcal{F}) \cong h^*(j_*\mathcal{F})$ naturally.

Thus let $X = \mathbb{P}^r_{\mathbb{C}}$ and let $\mathcal{G}$ be coherent $\mathcal{H}_X$-module on $\tilde{\mathbb{P}}^r_{\mathbb{C}} = X^{an}$.

By Cartan's Theorem B, there is a surjection:

$$\mathcal{H}_X^{M} \longrightarrow \mathcal{G}(m) \longrightarrow 0$$

for some $M, m \in \mathbb{Z}$. Thus $\mathcal{H}_X(-m)^{M} \longrightarrow \mathcal{G} \longrightarrow 0$.

Let R be the kernel of this surjection.

Then R is also a coherent $\mathcal{H}_X$-module so there is a

Surjection $\mathcal{H}_X^N(-n) \to \mathcal{R} \to 0$,

for some $n, N \in \mathbb{Z}$. Then we have

$$\mathcal{H}_X^N(-n) \xrightarrow{g} \mathcal{H}_X^M(-m) \longrightarrow \mathcal{G} \longrightarrow 0.$$

Now recall that $\mathcal{H}_X(-n)^N = (h^*\mathcal{O}_X(-n))^N = h^*(\mathcal{O}_X(-n)^N)$ and $\mathcal{H}_X(-m)^M = h^*(\mathcal{O}_X(-m)^M)$. Thus by fully faithfulness, $g = h^*f$ for $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-n)^N, \mathcal{O}_X(-m)^M)$.

Let $\mathcal{F} = \mathrm{Coker}(f)$. Then

$$\mathcal{O}_X(-n)^N \xrightarrow{f} \mathcal{O}_X(-m)^M \longrightarrow \mathcal{F} \longrightarrow 0 \quad \text{is}$$

exact, and hence so is

$$\mathcal{H}_X(-n)^N \xrightarrow{g} \mathcal{H}_X(-m)^M \longrightarrow \cancel{\mathcal{F}}\, h^*\mathcal{F} \longrightarrow 0$$

because h^* is exact and $h^*f = g$. Thus

$$\mathcal{G} \cong h^*\mathcal{F}.$$

∎

A few words about to prove Theorem I and II.

Theorem I:

- Step 1: Using $j : X \hookrightarrow \mathbb{P}^r_{\mathbb{C}}$ and that $H^q(X, \mathcal{F}) = H^q(\mathbb{P}^r_{\mathbb{C}}, j_* \mathcal{F})$, we reduce to $X = \mathbb{P}^r_{\mathbb{C}}$ (again using $j_*(h^* \mathcal{F}) \cong h^*(j_* \mathcal{F})$.)

- Step 2: Show directly for $\mathcal{F} = \mathcal{O}_X$, $h^* \mathcal{F} = \mathcal{H}_X$.

$$H^0(\mathbb{P}^r_{\mathbb{C}}, \mathcal{O}_X) \cong \mathbb{C} \cong H^0(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{H}_X)$$

$$H^q(\mathbb{P}^r_{\mathbb{C}}, \mathcal{O}_X) = 0 = H^q(\tilde{\mathbb{P}}^r_{\mathbb{C}}, \mathcal{H}_X), \quad q \geq 1$$

algebraic cohomology — Dolbeault's Theorem.

Step 3: Verify for $\mathcal{F} = \mathcal{O}_X(n)$ on $\mathbb{P}^r_{\mathbb{C}}$. This uses induction on the dimension r, and then induction on $|n|$. Assume $n > 0$. Then to induce the induction we take a hyperplane $E \cong \mathbb{P}^{r-1}_{\mathbb{C}}$. This gives rise to the LES:

use $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_E \to 0 \quad \otimes \quad \mathcal{O}(n)$:

$$
\begin{tikzcd}
H^{q-1}(E, \mathcal{O}_E(n)) \arrow[r] \arrow[dd, "\wr"'] & H^q(X, \mathcal{O}(n-1)) \arrow[r] \arrow[dd, "\wr"'] & H^q(X, \mathcal{O}(n)) \arrow[r] \arrow[dd, dashed, "\wr"'] & H^q(E, \mathcal{O}_E(n)) \arrow[d] \arrow[dd, bend right=60, "\wr"'] \\
 & & & H^q(X, \mathcal{O}(n-1)) \arrow[dd, bend left=60, "\wr"] \\
H^{q-1}(h^*E, \mathcal{H}_E(n)) \arrow[r] & H^q(X^{an}, \mathcal{H}_X(n-1)) \arrow[r] & H^q(X^{an}, \mathcal{H}_X(n)) \arrow[r] & H^q(h^*E, \mathcal{H}_E(n)) \arrow[d] \\
 & & & H^q(X^{an}, \mathcal{H}(n-1))
\end{tikzcd}
$$

and by induction and five lemma,

$$H^q(X, \mathcal{O}(n)) \xrightarrow{\sim} H^q(X^{an}, \mathcal{H}_X(n)).$$

<u>Step 4</u>: By Grothendieck vanishing, $H^q(X, \mathcal{F}) = 0$ for $q > n$, $n = \dim X$, X noetherian. Thus we can use descending induction on q.

Let $\mathcal{F}$ be coherent on X. Then $\mathcal{F}$ is quotient of $\mathcal{E} = \bigoplus_i \mathcal{O}(n_i)$, with kernel $\mathcal{N}$:

$$0 \to \mathcal{N} \to \mathcal{E} \to \mathcal{F} \to 0 \quad \text{gives}:$$

$$\begin{tikzcd}
H^q(X,N) \arrow[r] \arrow[d, "(1)"] & H^q(X,\mathcal{E}) \arrow[r] \arrow[d, "S"', "\text{by previous part}"] & H^q(X,\mathcal{F}) \arrow[r] \arrow[d, "(2)"] & H^{q+1}(X,N) \arrow[r] \arrow[d, "S"] & H^{q+1}(X,\mathcal{E}) \arrow[d, "S"] \\
H^q(X^{an},N^h) \arrow[r] & H^q(X^{an},\mathcal{E}^h) \arrow[r] & H^q(X^{an},\mathcal{F}^h) \arrow[r] & H^{q+1}(X^{an},N^h) \arrow[r] \arrow[r, phantom, "\text{by descending induction}"] & H^{q+1}(X^{an},\mathcal{E}^h)
\end{tikzcd}$$

Therefore by 5-lemma, middle map is surjective. Since we have shown surjectivity for arbitrary coherent $\mathcal{F}$, then (1) must be surjective also, and hence (2) is also injective. ∎

<u>Theorem II</u>: Let $X = \mathbb{P}^r_{\mathbb{C}}$ or $(\mathbb{P}^r_{\mathbb{C}})^h$, let $\mathcal{F}$ be coherent on X. Then for $n \gg 0$, $\mathcal{F}(n)$ is generated by global sections, i.e. $\mathcal{O}^N \twoheadrightarrow \mathcal{F}(n) \rightarrow 0$ for some N.

<u>proof</u>: (Analytic case is Cartan's Theorem A, which is harder and is true on any Stein space).

<u>Algebraic case</u>: $X = \operatorname{Proj} \mathbb{C}[x_0, \dots, x_r]$. Since $\mathcal{F}$ is coherent, then $\mathcal{F}|_{D_+(x_i)} \cong \tilde{M}_i$ where M_i is a f.g. $\mathbb{C}[\frac{x_0}{x_i}, \dots, \frac{x_r}{x_i}]$-module. Let $\{s_{ij}\}$ be a generating set for M_i (finite). Then there is an n s.t. $x_i^n s_{ij}$ extends to a global section t_{ij} of $\mathcal{F}(n)$. These t_{ij} are the generating sections for $\mathcal{F}(n)$. ∎

Applications of GAGA:

(1) GAGA gives a trivial proof that an analytic projective variety is algebraic (since the ideal sheaf is coherent). This is known as Chow's Lemma.

(2) GAGA gives a proof of the Riemann Existence Theorem:

Theorem: Let X be a smooth connected algebraic curve over $\mathbb{C}$. Then the following categories are equivalent:

(i) Finite étale covers of the variety X.
(ii) finite analytic covering maps of X
(iii) finite covering spaces of the topological space X.

GAGA proves the difficult step (ii) $\Rightarrow$ (i) as follows:

(1) GAGA induces an equivalence of categories between coherent algebras also, since the extra structure of being an algebra can be described in terms of module homs and commutative diagrams

(2) In this equivalence, generically separable $\mathcal{O}_X$-algebras correspond to generically separable $\mathcal{H}_X$-algebras, again because $\mathcal{O}_{X,\xi} \hookrightarrow \mathcal{H}_{X,\xi}$ is faithfully flat. Thus taking Spec we get an equivalence between branched algebraic covers and branched analytic covers.

(3) Finally, branched algebraic covers of X correspond to étale covers of an open n'hood of X.

Branched analytic covers correspond to analytic covers since analytic covers can always be extended.

This is of particular interest as it has applications to Galois Theory:

<u>Theorem</u>: Every finite group G is realizable as a Galois group over $\mathbb{C}(x)$.

<u>Ex</u>: Realise S_3 as G-cover of $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$.