Sheaf Cohomology on Schemes

Since $(X, \mathcal{O}_X)$ is a ringed space, then the category $\mathcal{O}_X$-mod has enough injectives, so given a quasi-coherent sheaf $\mathcal{F}$ on X, we have the derived functors $R^n\mathcal{F}$, which we denote as $H^n(X;\mathcal{F})$.

Recall that:

- $H^0(X;\mathcal{F}) = \mathcal{F}(X)$
- for every exact $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we have exact:

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to H^1(X;\mathcal{F}') \to H^1(X;\mathcal{F}) \to H^1(X;\mathcal{F}'') \to H^2(X;\mathcal{F}') \to \cdots$$

$H^n(X;\mathcal{F})$ is hard to compute, but we have two general results:

<u>Theorem (Grothendieck Vanishing)</u>: Let X be a scheme (noetherian) of dimension n. Then $H^i(X;\mathcal{F}) = 0$ $\forall\, i > n$ and any sheaf of abelian groups $\mathcal{F}$.

<u>proof</u>: See Hartshorne (III, Thm 2.7). ∎

(local ~~acyc~~ acyclicity)

<u>Theorem</u>: Let $X = \operatorname{Spec} A$, A noetherian. Then for all quasi-coherent sheaves $\mathcal{F}$ on X, and for all $i > 0$, we have

$$H^i(X;\mathcal{F}) = 0.$$

<u>proof</u>: Let $M = \Gamma(X,\mathcal{F})$. Let $0 \to M \to I^\bullet$ be injective resolution. Then $0 \to \tilde{M} \to \tilde{I}^\bullet$ is exact a
$\tilde{I}^\bullet$ is flasque. Taking global sections gives $0 \to M \to I^\bullet$

which is exact!

This shows that our local models are sufficiently simple, like open balls in $\mathbb{R}^n$.

However, in many cases this cohomology is simply not computable.

We would like to create a simple, computable resolution of $\mathcal{F}$ that computes $H^n(X;\mathcal{F})$.

Here we see that <u>singular</u> and <u>de Rham</u> are off the table, because we want more general sheaves than constant. This leads to Čech.

Čech Cohomology

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of abelian groups on X.

Let $\mathcal{U} = (U_i)$ be an open cover of X.

For notation, let $\mathcal{U}_{i_0,\dots,i_p} = \mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_p}$.

We define a complex $C^\bullet(\mathcal{U}, \mathcal{F})$ of abelian groups by

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(\mathcal{U}_{i_0, \dots, i_p})$$

and $d^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$

$$(d\alpha)_{i_0, \dots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}} \Big|_{\mathcal{U}_{i_0, \dots, i_{p+1}}}$$

Def$\underline{\text{n}}$: The p^{th} <u>Čech</u> <u>cohomology</u> <u>groups</u> of $\mathcal{F}$, w.r.t. $\mathcal{U}$ are

$$\check{\mathcal{H}}^p(\mathcal{U}; \mathcal{F}) = h^p(C^\bullet(\mathcal{U}, \mathcal{F})).$$

Čech cohomology is highly computable, as the next example shows:

<u>Ex</u>: Let S^1 be the circle with usual topology and let $\mathbb{Z}$ be the constant sheaf. Let $\mathcal{U}$ and V be two opens with intersect at two disjoint intervals. Then

$$C^0 = \Gamma(\mathcal{U}, \mathbb{Z}) \times \Gamma(V, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$$

$$C^1 = \Gamma(\mathcal{U} \cap V, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$$

and $d^0 : C^0 \to C^1$ takes $\langle a, b \rangle \mapsto \langle b-a, b-a \rangle$.

Hence $\check{H}^0(\mathcal{U}, \mathbb{Z}) \cong \mathbb{Z}$ and $\check{H}^1(\mathcal{U}, \mathbb{Z}) \cong \mathbb{Z}$.

This worked because we chose are intersections to have no cohomology.

<u>Caveats</u>: • Čech cohomology depends on the cover $\mathcal{U}$. One can remove this by taking direct limit over all coverings, but then one loses computability.

• In general, Čech cohomology does not give a LES from a short one.

We can sheafify the Čech complex to get a resolution of $\mathcal{F}$, as follows:

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<\dots<i_p} f_*(\mathcal{F}|_{\mathcal{U}_{i_0,\dots,i_p}})$$

where $f: V \hookrightarrow X$ is the open immersión.

That is,
$$\mathcal{C}^p(\mathcal{U}, \mathcal{F})(V) = \prod_{i_0<\dots<i_p} \mathcal{F}(\mathcal{U}_{i_0,\dots,i_p} \cap V).$$

We define d^p analogously as before.

<u>Theorem</u>: Taking global sections of the sheaf complex $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ we get the usual Čech complex $C^p(\mathcal{U}, \mathcal{F})$.

It is easy to see that if $\mathcal{F}$ is a sheaf then $\check{H}^0(\mathcal{U}; \mathcal{F}) = \mathcal{F}(X)$, (from the sheaf axioms).

Theorem: The complex

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U},\mathcal{F}) \to \mathcal{C}^1(\mathcal{U},\mathcal{F}) \to \cdots$$

is a resolution of $\mathcal{F}$.

proof: Hartshorne. III, Lemma 4.2 ∎

When does Čech compute the derived functor cohomology? It does this when we choose our cover simply enough.

Defn: An open covering $\mathcal{U} = (\mathcal{U}_i)$ is called Leray for $\mathcal{F}$ if for each $U = U_{i_0} \cap \cdots \cap U_{i_p}$, we have $H^k(U, \mathcal{F}|_U) = 0 \quad \forall \; k > 0$.

The big result for Čech cohomology is:

<u>Key Theorem</u>: If $\mathcal{U}$ is a Leray cover for $\mathcal{F}$, then the natural morphism

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F})$$

is an isomorphism.

<u>proof</u>: <u>Step 1</u>: Suppose $\mathcal{F}$ is flasque. Then

$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0 \quad \forall \ p > 0$. So they agree on

flasque sheaves.

This follows because the resolution $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ becomes a flasque resolution of $\mathcal{F}$, since each $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is a restriction / product of flasque sheaves.

<u>Step 2</u>: For general $\mathcal{F}$ we proceed by induction.

<u>$p=0$</u>: Then $\check{H}^0(\mathcal{U}, \mathcal{F}) \simeq \mathcal{F}(X) \simeq H^0(X, \mathcal{F})$.

For general p, embed $\mathcal{F} \to \mathcal{G}^*$ into an injective (and therefore flasque) sheaf. Let $\mathcal{R}$ be the coker.

Then

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0 \qquad (*).$$

is an exact sequence of sheaves.

Let U_α be a Čech open set. Then we get a LES

$$0 \to \mathcal{F}(U_\alpha) \to \mathcal{G}(U_\alpha) \to \mathcal{R}(U_\alpha) \to H^1(U_\alpha, \mathcal{F}|_{U_\alpha}) \to \dots$$

by acyclicity we get a SES

$$0 \to \mathcal{F}(U_\alpha) \to \mathcal{G}(U_\alpha) \to \mathcal{R}(U_\alpha) \to 0.$$

Taking the various products over α we get an exact:

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{R}) \to 0$$

of cochain complexes. By the general theory of homological algebra, this gives a LES of Čech cohomology.

Since $\mathcal{G}$ is flasque, by Step 1 we have that $\check{H}^q(\mathcal{U}, \mathcal{G}) = 0 \;\; \forall \; q > 0$ and so the LES decomposes into:

$$0 \to \check{H}^0(\mathcal{U},\mathcal{F}) \to \check{H}^0(\mathcal{U},\mathcal{G}) \to \check{H}^0(\mathcal{U},\mathcal{R}) \to \check{H}^1(\mathcal{U},\mathcal{F}) \to 0$$

and $\check{H}^p(\mathcal{U},\mathcal{R}) \cong \check{H}^{p+1}(\mathcal{U},\mathcal{F}) \quad \forall \quad p > 0$.

Hitting (*) with the LES of sheaf cohomology and using that $\mathcal{G}$ is flasque, we get

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\wr"] & \check{H}^0(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \check{H}^0(\mathcal{U},\mathcal{R}) \arrow[r] \arrow[d, "\wr"] & \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] & H^0(X,\mathcal{R}) \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

and so $\check{H}^1(\mathcal{U},\mathcal{F}) \simeq H^1(X,\mathcal{F})$. So true for $p=1$.

Now we have

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^p(\mathcal{U},\mathcal{R}) \arrow[r] \arrow[d] & \check{H}^{p+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^p(X,\mathcal{R}) \arrow[r] & H^{p+1}(X,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

So we have reduced to showing that

$$\check{H}^p(\mathcal{U},\mathcal{R}) \simeq H^p(X,\mathcal{R}) \quad \forall \ p \geq 1.$$

Step 3: We prove the general fact that if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$ is exact, and

$\mathcal{U}$ is Leray for $\mathcal{F}'$ and $\mathcal{F}$, then $\mathcal{U}$ is Leray for $\mathcal{F}''$.

Indeed, by the LES we get that $H^p(U_\alpha, \mathcal{F}''|_{U_\alpha}) = 0$ $\forall\ p \geq 1$.

In our case, $\mathcal{U}$ is Leray for $\mathcal{F}$, and is Leray for $\mathcal{G}$ because $\mathcal{G}$ is flasque. Hence $\mathcal{U}$ is Leray for $\mathcal{R}$ and so by induction hypothesis,

$$\check{H}^p(\mathcal{U}, \mathcal{R}) \simeq H^p(X, \mathcal{R}) \qquad p > 1.$$

which concludes the proof ∎

This result is not only beautiful, but it is extremely useful, As the next two theorems show.

pto.

Theorem: Let X be a noetherian, separated scheme, and let $\mathcal{U}$ be any affine open cover. Then

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}) \quad \forall p \geq 0$$

for all quasi-coherent sheaves $\mathcal{F}$ on X.

Theorem: (Cartan theorem B): If X is a stein space and $\mathcal{F}$ a coherent analytic sheaf, then

$H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1$.

Since any complex analytic space can be covered by stein spaces (and $\mathbb{C}^n$ is stein) we get that:

Theorem: Let X be a proj algebraic variety over $\mathbb{C}$, let $\mathcal{U}$ be the standard affine covering. Then

$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$ and $\check{H}^p(\mathcal{U}^{an}, \mathcal{F}^h) = H^p(X^{an}, \mathcal{F}^h)$

$\forall p \geq 0$, and $\mathcal{F}$ a coherent sheaf on X.