Sheaf Cohomology

Cohomology is a measure of the obstruction for local things to "glue" to give global things. Often these local-to-global problems can be asked in terms of sheaves, so we would like a theory of sheaf cohomology.

To have a local solution, we might say that we have an *exact* sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves on a space, and asking for a global solution is to ask: is

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$

also exact?

In general, the functor $\Gamma(X,-)$ is only left exact:

$$0 \to \Gamma(X,\mathcal{F}') \to \Gamma(X,\mathcal{F}).$$

We want our cohomology theory to measure what goes wrong on the right:

Abelian Categories and Injectives

An *abelian category* $\mathcal{C}$ is a category in which we can form $\ker f$ and $\operatorname{coker} f$ (and therefore exact sequences), among other things.

Ex: • R-mod, the category of modules over a ring R is the prototypical example.

• $(X,\mathcal{O})$ is a ringed space, then $\mathcal{O}$-mod, the category of sheaves of $\mathcal{O}$-modules is abelian.

Since the functor $\Gamma(X,-)$ is left exact, it will give rise to a theory of cohomology, (right exact $\to$ homology)

<u>Def</u>: In any abelian category $\mathcal{C}$, an object I is called <u>injective</u> if $\mathrm{Hom}(-, I)$ is an exact functor.

Here are some properties of injectives:

<u>Prop</u>: (1) A is injective if and only if $\forall\ i: B \hookrightarrow C$ each $\beta: B \to A$ extends to $\gamma: C \to A$ s.t. $\gamma \circ i = \beta$.

(2) Every $i: A \hookrightarrow B$, A injective, splits.

(3) If $0 \to A \to B \to C \to 0$ is exact and A injective, then B is injective iff C is injective.

Property (1) is important because it will show that injective objects are "resolvent" objects for a left exact functor (projective objects have a dual property that makes them "resolvent" for right exact functors).

<u>Resolutions</u>

Let A be an object in $\mathcal{C}$ and let ϕ be a subclass of objects in $\mathcal{C}$. A <u>resolution</u> of A

in $\mathcal{I}$ is an exact sequence

$$0 \to A \xrightarrow{\overset{\varepsilon}{\partial^{-1}}} I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} \cdots$$

where each $I^i \in \mathcal{I}$.

Given a resolution of an object A and (in $\mathcal{I}$) a left exact functor $\mathcal{F}$, we may form the complex

$$0 \to \mathcal{F}(A) \xrightarrow{\varepsilon} \mathcal{F}(I^0) \xrightarrow{\partial^0} \mathcal{F}(I^1) \xrightarrow{\partial^1} \cdots$$

and define $H^n(A, \mathcal{F}) = \dfrac{\ker \partial^n}{\operatorname{Im} \partial^{n-1}}$, the ~~sheaf~~ derived functor cohomology, i.e. the cohomology of A with values in $\mathcal{F}$. Also denoted $R^n\mathcal{F}(A)$, the <u>right derived functors.</u>

For this to be well defined, we need to know some things:

(A) • Every object $A \in \mathcal{C}$ has a resolution in $\mathcal{I}$

(B) • The cohomology does <u>not</u> depend on the resolution.

(C) • We'd also want that $R^n\mathcal{F}$ give a long exact sequence.

Let ϕ be the class of injective objects in C.

<u>Defn</u>: $\forall A \in C$, if $\exists A \hookrightarrow I$ with $I \in \phi$ then C <u>has enough</u> injectives.

<u>Ex</u>: Ab, abelian groups, has enough injectives.

An abelian group G is injective $\iff G$ is divisible.

Let $\mathbb{Z}(G)$ be the free abelian group generated by G.

There is a map $\mathbb{Z}(G) \xrightarrow{f} G$, $\sum_i n_i g_i \mapsto \sum_i n_i g_i$ and let $K = \ker f$. Take $\mathbb{Q}(G)$ be the free $\mathbb{Q}$-module with basis G and let $I(G) = \mathbb{Q}(G)/K$. Then $I(G)$ is divisible and $G \hookrightarrow I(G)$...

<u>Ex</u>: R-mod has enough injectives.

Let $M \in R$-mod. Then $M \in Ab$, so construct $I(M)$. Then the R-module is $\mathrm{Hom}_{\mathbb{Z}}(R, I(M))$ is injective and $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I(M))$, $m \mapsto (r \mapsto rm)$.

Fact: $\mathcal{O}_X$-mod has enough injectives.

Theorem: Let ϕ be the class of injective objects in $\mathcal{C}$, and assume $\mathcal{C}$ has enough injectives. Then

(a). Every object has a resolution in ϕ

(b). $R\mathcal{F}^n$ is well defined and is the n^{th} cohomology of a resolution in ϕ.

(c). We get LES.

proof: (a) follows from above.

(b) follows from properties of injective objects, use these to construct a chain homotopy.

(c) LES comes from standard diagram chase. $\blacksquare$

In this case we say that the injective ~~modules~~ objects are a resolvent family for $\mathcal{C}$. $\mathcal{F}$.

Theorem: Injective objects are $\mathcal{F}$-acyclic.

proof: If I is injective, then we can use the resolution $0 \to I \to I \to 0$, to compute $R^n \mathcal{F}(I)$. Then $0 \to \mathcal{F}(I) \to \mathcal{F}(I) \to 0$ is exact and so $R^n \mathcal{F}(I) = 0 \quad \forall\, n > 0$. ∎

When $\mathcal{C}$ has enough injectives, then the derived functors $R^n \mathcal{F} = H^n(-,\mathcal{F})$ are called cohomology of $\mathcal{F}$ and are the universal solution satisfying

1) $H^0(X,\mathcal{F}) = \mathcal{F}(X)$

2) If $0 \to A' \to A \to A'' \to 0$ is SES

then $\exists$ LES

$$0 \to A' \to A \to A'' \to H^1(A',\mathcal{F}) \to H^1(A,\mathcal{F}) \to H^1(A'',\mathcal{F}) \to H^2(A',\mathcal{F}) \to \cdots$$

<u>Key Lemma</u>: Let $A \to J^\bullet$ be a resolution of A by $\mathcal{F}$-acyclic objects. Then

$$R^n\mathcal{F}(A) \xrightarrow{\sim} \mathcal{H}^n(\mathcal{F}(J^\bullet)).$$

<u>proof</u>: Let $K^n = \ker\{J^n \to J^{n+1}\}$.

$$0 \to K^p \to J^p \to K^{p+1} \to 0 \qquad p \geq 0$$

$$\Rightarrow \quad R^q\mathcal{F}(K^p) \simeq R^{q-1}\mathcal{F}(K^{p+1}), \quad p \geq 0, q \geq 1$$

and $$R^1\mathcal{F}(K^{n-1}) \simeq \frac{\mathcal{F}(K^n)}{\mathrm{Im}(\mathcal{F}(J^{n-1}) \to \mathcal{F}(K^n))} \simeq \mathcal{H}^n(J).$$

Hence by induction $R^n\mathcal{F}(A) \simeq \mathcal{H}^n(J)$ ∎

In practice, we never use injective resolutions to compute cohomology, instead we use the key lemma to find an acyclic resolution!

Def$^{\underline{n}}$: Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules. Then

(a) $\mathcal{M}$ is called <u>flasque</u> (flabby) if $\forall$ open $\mathcal{U} \subset X$, the restriction map $\Gamma(X, \mathcal{M}) \to \Gamma(\mathcal{U}, \mathcal{M})$ is surjective.

(b) $\mathcal{M}$ is called <u>soft</u> if for every closed $Y \subset X$, the restriction map $\Gamma(X, \mathcal{M}) \to \Gamma(Y, \mathcal{M})$ is surjective.

(c) $\mathcal{M}$ is called <u>fine</u> if $\forall$ locally finite open cover $\{\mathcal{U}_i\}$ of X there is a family $\{\phi_i : \mathcal{M} \to \mathcal{M}\}_i$ such that ϕ_i is supported in $\mathcal{U}_i$ and $\sum_i \phi_i = \mathrm{Id}$.

The family $\{\phi_i\}$ is a partition of unity of $\mathcal{M}$.

We will see that in "nice" situations, these sheaves are acyclic and so maybe used to compute derived functor sheaf cohomology.

Theorem: Let X be a ringed top. space, Let $\mathcal{C} = \mathcal{O}_X$-mod. $\mathcal{F} = \Gamma(X, -)$.

(i) Injective $\Longrightarrow$ Flasque $\Longrightarrow$ $\mathcal{F}$-acyclic

(ii) If X is paracompact then

$$\begin{array}{ccc} \text{flasque} & \searrow & & \\ & & \text{soft} \Longrightarrow \mathcal{F}\text{-acyclic} \\ \text{fine} & \nearrow & & \end{array}$$

proof: omitted.

Ex: Let C^0 be the sheaf of real valued continuous functions on X. If X is paracompact then C^0 is fine.

Ex: If X is a C^p-differentiable manifold, then the sheaves C^p are fine

Ex: Let M be a C^∞-manifold. Let Ω^p be the sheaf of C^∞-differential forms on M of degree p. The poincaré lemma shows that

$$0 \to \mathbb{R} \to \mathcal{E} \to \Omega^1 \to \Omega^2 \to \cdots$$

is an exact sequence of sheaves, where $\mathbb{R}$ is the sheaf of locally constant $\mathbb{R}$-valued functions. This is a resolution of $\mathbb{R}$ by fine sheaves, and so

Theorem (De Rham):

$$H^n_{dR}(X;\mathbb{R}) \simeq H^n(X;\mathbb{R}) \quad \forall\, n \geq 0.$$

Ex: Let X be a CW-complex (paracompact) and let $\mathbb{Z}$ be the constant sheaf on X. Then

$$0 \to \mathbb{Z} \to S^1(X;\mathbb{Z}) \to S^2(X;\mathbb{Z}) \to \cdots$$

is a soft resolution of $\mathbb{Z}$, where $S^p(X;\mathbb{Z})$ denotes the "sheafified" sheaf of p-cochains with integer coefficients.

Thus

$$H^n_{sing}(X;\mathbb{Z}) \simeq H^n(X;\mathbb{Z})$$

This shows that derived functors are the "true" cohomology theory, and for sufficiently nice spaces and sheaves, we can find explicit resolutions to compute it.

<u>Applications</u>

Coherent sheaf cohomology on Schemes via derived functors and Čech