Spectral Sequences

(K, D) a differential complex with filtration $\{K_p\}$.

Define $A = \bigoplus_{p\in\mathbb{Z}} K_p$ and $GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}$.

Define $i : A \longrightarrow A$ by using $K_{p+1} \hookrightarrow K_p$ $\forall p$. Hence $i(K_{p+1}) \subset K_p$. We have a SES

$$0 \longrightarrow A \xrightarrow{i} A \xrightarrow{j} B \longrightarrow 0 \qquad (*)$$

where B is the cokernel of i.

Then $B = A/\mathrm{Im}(i) = \dfrac{\bigoplus K_p}{\bigoplus K_{p+1}} = GK$.

In $(*)$, each group is a complex with differential induced from D.

If K is graded (i.e. a cochain complex) then $(*)$ becomes a SES of cochain complexes and we get a LES of Cohomology groups

$$\to H^k(A) \xrightarrow{i_1} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \to$$

which we write as

$$\begin{tikzcd} H^*(A) \arrow[rr, "i_1"] & & H^*(A) \arrow[dl, "j_1"] \\ & H^*(B) \arrow[ul, "k_1"] & \end{tikzcd}$$

This is an exact couple, and so it gives rise to a sequence of exact couples

$$\begin{tikzcd} A_r \arrow[rr, "i"] & & A_r \arrow[dl, "j_r"] \\ & E_r \arrow[ul, "k_r"] & \end{tikzcd}$$

for each r, each being derived from the previous one.

Ex: Suppose K has a filtration

$$K = K_0 \supset K_1 \supset K_2 \supset K_3 \supset 0.$$

Then $A_1 = H^*(A) = H^*(K_0 \oplus K_1 \oplus K_2 \oplus K_3)$

$$= \bigoplus_{p=0}^{3} H^*(K_p)$$

and so is the direct sum of all the terms in the sequence:

$$H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow 0$$

which is not exact.

Next, A_2 is the image of A_1 under i, and so is the direct sum of the terms:

$$H(K) \supset iH(K_1) \leftarrow iH(K_2) \leftarrow iH(K_3) \leftarrow 0.$$

Note here that $iH(K_1) \subset H(K)$ is an inclusion.

We repeat this for A_3 and A_4 and observe that

A_4 is the sum of

$$H(K) \supset iH(K_1) \supset iiH(K_2) \supset iiiH(K_3) \supset 0 \quad (+)$$

which are all inclusions.

Since we have inclusions, the derived procedure stabilizes and we get $A_4 = A_5 = A_6 = \cdots = A_\infty$.

Further, since

$$\begin{tikzcd} A_4 \arrow[rr, "i"] & & A_4 \arrow[dl] \\ & E_4 \arrow[ul, "k_4"] & \end{tikzcd}$$

is exact, and i injective, then k_4 is 0.

Therefore, after the 4^{th} derivation, all the differentials of the exact couple are 0 and therefore $E_4 = E_5 = \cdots = E_\infty$.

Since E_∞ is a quotient of i_∞, it is a direct sum of successive quotients in i_∞. If $(+)$ is the filtration on $H(K)$, then $E_\infty = GH(K)$ under $(+)$.

In general, the filtration $K \supset K_1 \supset K_2 \supset K_3 \supset \ldots$

induces a sequence in cohomology:

$$\leftarrow H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow \ldots$$

where i is not inclusion. Let $F_p = \mathrm{Im}\left(H(K_p) \xrightarrow{i^{p}} H(K)\right)$.

Then we get the induced filtration on $H(K)$:

$$H(K) \supset F_1 \supset F_2 \supset F_3 \supset \ldots$$

If the filtration $\{K_p\}$ has finite length, say l,

then A_l and E_l are stationary and the value

$E_\infty = \bigoplus F_p / F_{p+1}$, the associated graded

cohomology of $H(K)$.

The terms E_r are a <u>spectral sequence</u>.

<u>Theorem</u>: Let $K = \bigoplus_{n \in \mathbb{Z}} K^n$ be a graded filtered complex with filtration $\{K_p\}$ and let $H_D^*(K)$ be the cohomology of K with induced filtration.
If for each dimension n, the filtration $\{K_p^n = K^n \cap K_p\}$ has finite length, then the short exact sequence

$$0 \longrightarrow \bigoplus K_{p+1} \longrightarrow \bigoplus K_p \longrightarrow \bigoplus K_p/K_{p+1} \longrightarrow 0$$

induces a spectral sequence which converges to $H_D^*(K)$, i.e. $E_\infty = GH_D^*(K)$.

<u>proof</u>: Let $A_r = \bigoplus_{p \in \mathbb{Z}} i^{r-1} H(K_p)$. If $r \geq p+1$, then $i^r H(K_p) = F_p$ and $i : i^r H(K_{p+1}) \longrightarrow i^r H(K_p)$ is an inclusion.
Now on each derived couple, i and j preserve dimension but k increases by 1 (because of LES).
Given n, let $\ell(n)$ be the length of $\{K_p^n\}$ and let $r \geq \ell(n+1)+1$.

Then for any integer p,

$$i^r H^{n+1}(K_{p+1}) = F_{p+1}^{n+1} \quad \text{and}$$

$$i : i^r H^{n+1}(K_{p+1}) \longrightarrow i^r H^{n+1}(K_p)$$

is an inclusion. Hence $i_r : A_r^{n+1} \longrightarrow A_r^{n+1}$ is an inclusion and $k_r : E_r^n \longrightarrow A_r^{n+1}$ is 0. Hence E_r^n is stationary and E_∞^n is this stationary value.

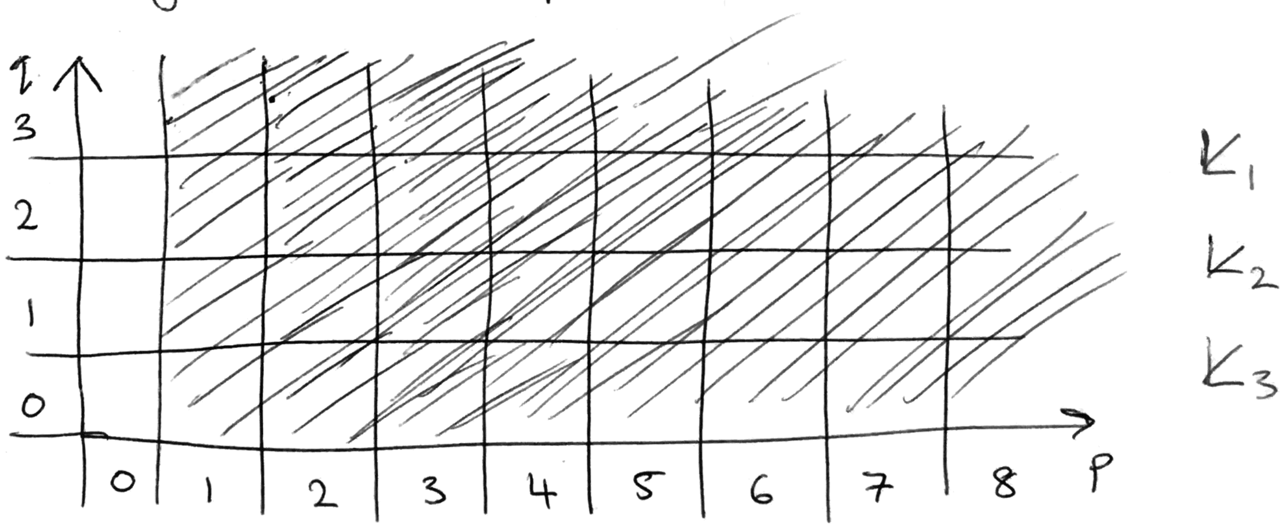
Note that $A_\infty^n = \oplus F_p^n$ and $i_\infty : F_{p+1}^n \rightarrow F_p^n$ for every n. Since $i_\infty : \oplus F_{p+1} \hookrightarrow \oplus F_p$ is inclusion, then $E_\infty = \oplus F_p / F_{p+1} = GH_p^*(K)$. $\blacksquare$

Let $K = \bigoplus K^{p,q}$ be a double complex with horizontal operator δ and vertical operator d. We form a single complex by letting

$$C^k = \bigoplus_{p+q=k} K^{p,q}$$

$$D = \delta + (-1)^p d : C^k \to C^{k+1}.$$

Then $K = \bigoplus_k C^k$. We get a filtration on K by setting $K_p = \bigoplus_{i \geq p} \bigoplus_{q \geq 0} K^{i,q}$:



$A = \bigoplus K_p$ is also a double complex, which we turn into a single complex $A = \bigoplus A^k$ by summing bidegrees, so A^k consists of elements of A whose total degree is k.

There is an inclusion $i : A^k \longrightarrow A^k$ given by

$$i : A^k \cap K_{p+1} \hookrightarrow A^k \cap K_p .$$

The single complex A inherits the operator D from K.

Similarly, $E = \bigoplus K_p / K_{p+1}$ can be made into a single complex with operator D. Because $\delta : K_p \to K_{p+1}$, then D on E is just $(-1)^p d$. Hence

$$E_1 = H_D(E) = H_d(K),$$

Cohomology of K w.r.t. d.