Function Fields of Integral Schemes

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We introduce the basic theory of function fields of schemes via valuations on a field. We then apply our theory to study global sections of schemes analogous to projective varieties.

1 Valuations

We build the needed theory of valuations. Our main reference will be [1]. We fix R to be an integral domain with field of fractions K. We call the pair $(A, \mathfrak{m}_A)$ a local ring if A is a ring with unique maximal ideal $\mathfrak{m}_A$.

Definition 1.1. We call R a *valuation ring (of K)* if for all $x \in K$ either $x \in R$ or $x^{-1} \in R$. The case $R = K$ is the *trivial* valuation ring. A valuation ring R is called a *discrete valuation ring* (DVR) if R is a PID.

If R is a valuation ring, then for any two proper ideals I, J of R we must have either $I \subset J$ or $J \subset I$. This follows since if $x \in I$ and $x \notin J$, then for any $0 \neq y \in J$, we have $x/y \notin R$, so that $y/x \in R$. But then $y = (y/x) \cdot x \in I$. Thus $J \subset I$.

Thus the proper ideals of R are totally ordered and so R is local with unique maximal ideal $\mathfrak{m}$. We also observe that $K - R = \{x \in K^\times \mid x^{-1} \in \mathfrak{m}\}$ and so R is determined by K and $\mathfrak{m}$.

Definition 1.2 Let K/k be a field extension. An integral domain R is said to be a *valuation ring of* K/k if $k \subset R \subset K$ and R is a valuation ring of K.

Definition 1.3. Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be local rings. We say that A *dominates* B if $B \subseteq A$ and $\mathfrak{m}_A \cap B = \mathfrak{m}_B$.

Using domination defined above, one can partially order the local subrings of a field, and thus speak about a *maximal* local subring with respect to this relation. We state some facts about valuation rings. All proofs for the this section are omitted and given the appropriate reference.

Theorem 1.4. *A local ring R of a field K is a valuation ring if and only if R is a maximal local subring of K with respect to the domination relation. Every local subring of K is dominated by some valuation ring of K.*

Proof. See [[2], Ch. 5, p. 65]. □

Definition 1.5. Let K be a field and $A \subset K$ a subring. We say that a valuation ring $(R, \mathfrak{m}_R)$ of K has *center in* A if $A \subset R$ and we call the prime ideal $\mathfrak{m}_R \cap A$ the *center of* R in A.

It turns out that we can view valuation rings of field extensions as 'points' and in a particular case, these points fit together smoothly to make a nonsingular projective curve.

Definition 1.6. Let K be a field and $A \subset K$ a subring. The *Zariski-Riemann Space* $Z(K, A)$ is the set of all valuation rings of K with center in A.

One can make $Z(K, A)$ into a topological space by endowing it with the cofinite topology, from which it follows that $Z(K, A)$ is quasi-compact (see [[1], Ch. 4, p. 73-74]). Now suppose k is an algebraically closed field and K/k a field extension of transcendence degree 1. Then the subspace $Y \subset Z(K, k)$ of discrete valuation rings is isomorphic to an abstract nonsingular curve over k, that is, a abstract nonsingular projective variety of dimension 1 over k. This gives an equivalence of categories:

$$\text{Nonsingular Projective Curves over } k$$
$$\updownarrow$$
$$\text{Field extensions } K/k \text{ of transcendence degree } 1$$

For the above see [[3], Ch. 1, Sec. 6].

2 Motivation

Our motivation for the definition of function field of schemes comes from varieties. Let X be an algebraic variety over an algebraically closed field k with function field $k(X)$, then

$$k(X) \cong A(X)_{((0))}$$

where $A(X)$ is the coordinate ring of X and (0) is the ideal of the zero polynomial.

For $A(X)_{((0))}$ to be a field, we require $A(X)$ to be an integral domain. Hence we will only be able to define the function field of a particular class of schemes, ones whose structure sheaf gives integral domains.

Definition 2.1. A scheme $(X, \mathcal{O}_X)$ is called *integral* if for each open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

3 Affine Schemes

Since our definition of function field for a scheme involves 'local' constructions, we prove some results for affine schemes. Then our global results will follow by patching these results together.

Let $X = \text{Spec}A$ be an affine scheme. Then X being integral implies that A is an integral domain. It is a fact that a closed subset $Y \subset X$ is irreducible if and only if $I(Y) \subset A$ is a prime ideal.

We want to find a connection between irreducible sets and generic points.

Definition 3.1. Let $Y \subset X$ be a closed subscheme of X. A point $y \in Y$ is called a *generic point* if $\bar{\{y\}} = Y$.

We define the notation that if $x \in \text{Spec}A$, we write $\mathfrak{p}_x$ to mean x as a prime ideal of A, rather than a point of $\text{Spec}A$. Then we see that

$$\bar{\{x\}} = V(I(\{x\})) = V(\sqrt{\mathfrak{p}_x}) = V(\mathfrak{p}_x)$$

for any point $x \in X$. Hence every closed irreducible subset $Y \subset X$ has a unique generic point.

Furthermore, $\text{Spec}A$ has a generic point x if and only if $\mathfrak{p}_x \subset \text{Nil}(A)$, the nilradical of A. Since A is an integral domain, it follows that (0) is the unique generic point of $\text{Spec}A$. Finally we have that $\mathcal{O}_{\text{Spec}A,(0)} = A_{(0)}$ which is a field since A is an integral domain.

4 Function Field of a Scheme

We can only define the function field for integral schemes (see Definition 2.1 above). Before we can define the function field, we need some results. The main reference for the rest of the notes is [3].

Lemma 4.1. *Let $(X, \mathcal{O}_X)$ be an integral scheme. Then*

(1) *X is irreducible.*

(2) *Every closed and irreducible subset of X has a unique generic point.*

Proof. For (1). Suppose to the contrary that $X = K_1 \cup K_2$ for proper, nonempty closed subsets K_1, K_2. Letting $U_1 = X - K_1$ and $U_2 = X - K_2$ we have U_1, U_2 are nonempty open subsets such that $U_1 \cap U_2 = \emptyset$. But then $\mathcal{O}(U_1 \cup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$, the right hand side of which cannot be an integral domain.

Now for (2). Let $Y \subset X$ be closed and irreducible. Choose an affine open subset $U \cong \text{Spec}A$ of X such that $Y \cap U \neq \emptyset$. Then $U \cap Y$ is closed and irreducible inside U, and therefore by Section 3, we have that $U \cap Y$ has a unique generic point y. But then it follows that $\bar{\{y\}} = Y$ since $U \cap Y$ is open inside of Y, and therefore is dense inside of Y. □

It follows from the previous lemma that every integral scheme has a unique generic point.

Lemma 4.2. *Let $(X, \mathcal{O}_X)$ be an integral scheme with generic point ξ. Then the stalk $\mathcal{O}_{X,\xi}$ is a field.*

Proof. Let $U \cong \text{Spec}A$ be an affine open subset of X containing ξ. Since the local ring $\mathcal{O}_{X,\xi}$ is 'local', it is equal on smaller open sets, that is $\mathcal{O}_{X,\xi} = \mathcal{O}_{U,\xi}$. Hence we can assume that X is affine. But then by Section 3 we have that

$$\mathcal{O}_{X,\xi} = \mathcal{O}_{U,\xi} \cong A_{(0)}$$

which is a field since A is an integral domain. □

Definition 4.3. Let $(X, \mathcal{O}_X)$ be an integral scheme with generic point ξ. The *function field* $k(X)$ of X is the stalk $\mathcal{O}_{X,\xi}$.

Remark. The proof of Lemma 4.2 did not depend on the affine open set chosen. Hence we see that the function field of X is equal to the fraction field of A for any affine open set Spec$A$ of X.

5 Main Result

We combine the theory of valuations and function fields to prove the following result:

Main Theorem. *Let k be an algebraically closed field and $(X, \mathcal{O}_X)$ an integral proper scheme over k. Then $\Gamma(\mathcal{O}_X, X) \cong k$.*

This result generalizes the fact that *the only globally defined regular functions on a projective variety are the constant functions.*

Before we can prove the theorem, we must define the assumptions made in the theorem, that is, we must define what it means for a scheme to be *separated, proper* and *finite type over a field.*

Definition 5.2. A scheme X is of *finite type over a field k* if there exists an affine open cover $\{\text{Spec}A_i\}_i$ of X such that each A_i is a finitely generated k-algebra.

Definition 5.3. A scheme X is *separated* over k if the diagonal map $X \to X \times_{\text{Spec}k} X$ is a closed immersion.

Definition 5.4. A scheme X is *proper* over k if X is separated and of finite type over k, and X is *universally closed*: the map $f : X \to \text{Spec}k$ is closed and for any morphism $Y \to \text{Spec}k$ the induced morphism $f' : X \times_{\text{Spec}k} Y \to Y$ is closed.

We now interpret the notion of properness in terms of valuations. This criteria is due to Chevalley.

Theorem 5.5. (Valuative Criteria for Properness) *Let $f : X \to Y$ be a morphism of schemes of finite type. Then f is proper if and only if for every valuation ring R in a field K, and every morphism Spec$K \to X$ and Spec$R \to Y$ we have the following commuting diagram*

$$\begin{array}{ccc} \text{Spec}K & \longrightarrow & X \\ \downarrow & \overset{\exists! g}{\nearrow} & \downarrow f \\ \text{Spec}R & \longrightarrow & Y \end{array}.$$

We only require one implication to prove our main theorem so we only prove that direction here, and refer the reader to [[3], Ch. 2, p.101] for the rest of the proof. We also only require the existence and leave uniqueness to the reference given. In order to prove this theorem, we understand what it means to have morphisms Spec$K \to X$ and Spec$R \to Y$.

Lemma 5.6. *A morphism Spec$K \to X$ is the same as a choice of point $x \in X$ and an inclusion of fields $\mathcal{O}_{X,x}/\mathfrak{m}_x \to K$.*

A morphism Spec$R \to Y$ is the same as a choice of two points $y_0, y_1 \in Y$, with $y_0 \in \{\bar{y_1}\}$, and an inclusion of fields $\mathcal{O}_{Y,y_1}/\mathfrak{m}_{y_1} \to K$ such that R dominates $\mathcal{O}_{Z,y_0}$, where $Z = \{\bar{y_1}\}$.

Proof of Lemma 5.6. For the first statement. The scheme Spec$K$ is a point with sheaf K. Choosing $x \in X$ as the image, then to be a morphism we must have a local ring homomorphism $\mathcal{O}_{X,x} \to K$, which to be local means $\mathfrak{m}_x \mapsto 0$. Hence the ring homomorphism factors into an injective map $\mathcal{O}_{X,x}/\mathfrak{m}_x \to K$. Clearly given this information we can define a morphism.

For the second statement. The scheme Spec$R$ is two points t_0, t_1, one closed and one generic (since R is a valuation ring) respectively. Let y_0, y_1 be there respective images in Y. Since our morphism is continuous, it follows that $y_0 \in Z = \{\bar{y_1}\}$. Hence we get a map Spec$R \to Z$. The function field of Z is the same as $\mathcal{O}_{Y,y_1}/\mathfrak{m}_{y_1}$. Hence if we have a morphism, we must get an inclusion of fields $\mathcal{O}_{Y,y_1}/\mathfrak{m}_{y_1} \to K$ and a local ring homomorphism $\mathcal{O}_{Z,y_0} \to R$. That is, R dominates $\mathcal{O}_{Z,y_0}$. Conversely, given the data, the inclusion $\mathcal{O}_{Z,y_0} \to R$ gives a morphism Spec$R \to$ Spec$\mathcal{O}_{Z,y_0}$, which when we compose with the natural map Spec$\mathcal{O}_{Z,y_0} \to X$ gives the required morphism. □

Now we can prove Theorem 5.5 (the valuative criteria for properness) using the lemma.

Proof of Theorem 5.5. Assume the diagram with the morphism f proper. Let $X' = X \times_Y$ Spec$R$. By composition we get a map Spec$K \to X'$. Let ξ_1 be the image of the point in Spec$K$ and let $Z = \{\bar{\xi_1}\}$. Then Z is closed in X'. Since f is proper, the morphism $f' : X' \to$ Spec$R$ is closed and so $f'(Z)$ is closed in Spec$R$. But $f'(\xi) = t_1$, the generic point of Spec$R$, and hence by continuity $f'(Z) =$ Spec$R$. Thus there exists $\xi_0 \in Z$ with $f'(\xi_0) = t_0$, the closed point of Spec$R$. We get a local ring homomorphism $R \to \mathcal{O}_{Z,\xi_0}$ induced by f'. That is, $\mathcal{O}_{Z,\xi_0}$ dominates R. Now the function field of Z is $\mathcal{O}_{X',\xi_1}/\mathfrak{m}_{\xi_1} \subset K$. Hence $\mathcal{O}_{Z,\xi_0} \subset K$. But R is maximal for domination of local subrings of K by Theorem 1.4. Hence $R \cong \mathcal{O}_{Z,\xi_0}$ and in particular must dominate it. But this is precisely the data for a morphism Spec$R \to X'$ by Lemma 5.6. Composing with the map $X' \to X$ gives the required morphism Spec$R \to X$. □

Combining Theorem 5.5 and the theory of valuations established in Section 1, we can prove Theorem 5.1.

Proof of Main Theorem. Let K be the function field of X and let $a \in \Gamma(\mathcal{O}_X, X)$. Assume $a \notin k$. Since k is algebraically closed, a must be transcendental over k. Since $\Gamma(\mathcal{O}_X, X) \subset K$ we know the fraction field of $\Gamma(\mathcal{O}_X, X)$ is also contained in K. Thus we can form the subring $k[a^{-1}] \subset K$ which must be a polynomial ring. By localizing $k[a^{-1}]$ at the maximal ideal (a^{-1}) and using Theorem 1.4, it follows that there exists a valuation ring R of K/k such that $k[a^{-1}] \subset R$ and $\mathfrak{m}_R \cap k[a^{-1}] = (a^{-1})$. Hence $a^{-1} \in \mathfrak{m}_R$. We claim that there must exist an $x \in X$ such that R dominates $\mathcal{O}_{X,x}$. This follows by the natural diagram

$$\begin{array}{ccc} \mathrm{Spec}K & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}R & \longrightarrow & \mathrm{Spec}k \end{array}.$$

By assumption, the morphism f is proper and hence by Theorem 5.5 there exists a morphism Spec$R \to X$. By Lemma 5.6, this morphism is equivalent to the data that there exists a point $x \in X$ such that R dominates $\mathcal{O}_{X,x}$ as required. Thus we have that $\mathfrak{m}_R \cap \mathcal{O}_{X,x} = \mathfrak{m}_x$. Since $a^{-1} \in \mathfrak{m}_R$, we have that $a \notin R$. Now $a \notin \mathfrak{m}_x$ since otherwise $a \in R$. But then a is a unit inside of $\mathcal{O}_{X,x}$ and hence $a^{-1} \in \mathcal{O}_{X,x}$. This gives $a^{-1} \in \mathfrak{m}_x$. Contradiction. Hence $a \in k$. $\square$

Remark. Finally we observe that a projective variety V/k is a proper, integral scheme of finite type over k ([[3],Ch.2,pg. 103-105]). When k is algebraically closed, we recover the well known result that the global regular functions on a projective variety are constant ([[3], Ch. 1, pg. 18]).

References

[1] Hideyuki Matsumura. *Commutative Ring Theory.* Cambridge University Press, 1989.

[2] M.F Atiyah and I.G Macdonald. *Introduction to Commutative Algebra.* Addison-Wesley, 1969.

[3] Robin Hartshorne. *Algebraic Geometry.* Springer, 1980.